A derivation of the relativistic Doppler effect

Consider 2 frames, S and S'. S' is moving with velocity $\vec{v} = +v_x$ with respect to S. An observer O sits at the origin of S, and an observer O' sits at the origin of S', moving with it.

At $t = t' = 0$, the origins coincide, and a transmitter sitting at rest at the origin of S starts emitting a signal. It emits a wave train (a long continuous signal) of N wavelengths and then stops. Note that O' is moving *away* from the transmitter.

For the observer O in the S frame: The front of the wave train reaches him at $x = 0, t = 0$. The end of the wave train passes him at $x = 0, t = t_f$ when N wavelengths have passed. Therefore, the frequency he observes is:

$$f_0 = \frac{N}{t_f} \tag{1}$$

We now *stay* in the S frame, but consider two *different* events. Event 1 is when the front of the wave train passes O'. Event 2 is when the end of the wave train passes O', i.e., when O' see N wavelengths pass him.

Event 1 is easy - the front passes him at $x_1 = 0, t_1 = 0$. As time passes, more and more of the wave train overtakes him. This is because at any given time t, the front of the wave train has reached $x_{signal\ front} = ct$, but O' has only reached $x_{O'} = vt < ct$. Similarly, the end of the wave train (signal end) is at $x_{signal\ end} = 0$ at $t = t_f$. At $t = t_2$, $x_{signal\ end} = c(t_2 - t_f)$. At the same time $t = t_2$, $x_{O'} = vt_2$.

We are looking for the time when the end of the wave train passes O', which will happen at that time when:

$$x_{O'} = x_{signal\ end}$$

which implies:

$$vt_2 = c(t_2 - t_f)$$

which gives:

$$t_2 = \frac{ct_f}{(c - v)} \tag{2}$$

Therefore, the second event, when the end of the wave train passes O', happens at these coordinates in S. To summarize; in the frame S, the two events are:

Event 1. The front of the wave train passes O' at

$$x_1 = 0$$

$$t_1 = 0$$

Event 2. The end of the wave train passes O' at

$$x_2 = vt_2$$

$$t_2 = \frac{ct_f}{(c - v)}$$

We use the Lorentz transformation equations

$$x' = \gamma(x - vt) \tag{3}$$

$$t' = \gamma(t - \frac{vx}{c^2}) \tag{4}$$

to get:

Event 1.

$$x'_1 = 0$$

$$t'_1 = 0$$

Event 2.

$$x'_2 = \gamma(vt_2 - vt_2) = 0$$

$$t'_2 = \gamma(t_2 - (\frac{v^2}{c^2} * t_2))$$

$$= t_2\gamma(1 - \frac{v^2}{c^2}) = \frac{t_2 * \gamma}{\gamma^2}$$

$$= \frac{t_2}{\gamma}$$

Which, using equation (2), gives:

$$t'_2 = \frac{ct_f}{(c - v)\gamma} = \frac{t_f}{(1 - \beta)\gamma}$$

$$= \frac{t_f * \sqrt{1 - \beta^2}}{1 - \beta}$$

$$= t_f * \sqrt{\frac{1 + \beta}{1 - \beta}}$$

Therefore, observer O', in the frame S', observes N wavelengths pass him in a time $t'_2 - t'_1$, which is:

$$\Delta t = t_f * \sqrt{\frac{1 + \beta}{1 - \beta}} \tag{5}$$

And so, he calculates the frequency to be:

$$f = \frac{N}{\Delta t} = \frac{N}{t_f} * \sqrt{\frac{1 - \beta}{1 + \beta}} \tag{6}$$

$$f = f_0 * \sqrt{\frac{1 - \beta}{1 + \beta}} \tag{7}$$

which is the relativistic Doppler effect for a source and observer when the two are receding from each other. You can see that $f < f_0$, i.e. it is *red* shifted, and that for increasing β, the observed frequency drops further and further below the original f_0. For the Doppler formula in the situation when the transmitter and observer are *approaching* each other, simply change β to $-\beta$, to get:

$$f = f_0 * \sqrt{\frac{1+\beta}{1-\beta}} \tag{8}$$

and we see that the observed frequency f will be higher than the original f_0, that is, it is *blue* shifted.